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Isotropic Elastic Solids of nearly Spherical Form.

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PART II.—VIBRATIONS.

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Section I.

§1. The present paper is complementary to a previous one* which dealt with the equilibrium of bodies of nearly spherical form. The method is practically the same, but the differences in detail are considerable. The treatment of the vibration problem is tantamount to assuming that answering to a natural type of vibration in a perfect sphere there is a very similar type of nearly equal frequency in a nearly spherical body. The principal object is to find what may be regarded as the change in pitch due to a small change in the shape of the surface; the result shows what effect an absence of perfect sphericity has on the frequency of vibrations. The present memoir thus does for irregularities in the shape of the surface what a previous memoir,* "On Some Compound Vibrating Systems," did for irregularities in the structure of the material.

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* American Journal of Mathematics, vol. 16, p. 299.

§2. The first thing required is the complete form of the displacements and stresses for the vibrations natural to a sphere. These I shall simply quote from a previous paper,† slightly altering its notation.

Let r, θ, ϕ denote the usual polar coordinates, θ answering to the colatitude, and ϕ to the longitude. Let Δ denote the dilatation, and u, v, w the displacements at the point (r, θ, ϕ) in the directions of the fundamental elements dr, $r\,d\theta$, $r \sin\theta d\phi$. Also for shortness let

$$\alpha^2 = \rho/(m+n),\ \beta^2 = \rho/n, \tag{1}$$

where ρ is the density and m, n the elastic constants in the notation of Thomson and Tait's Natural Philosophy.

Then as types of the dilatation and displacements we have, taking $k/2\pi$ as the frequency of vibration,

$$\Delta = \cos kt\, r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\alpha r)\, Y_i\mathbf{Y}_i, \tag{2}$$

$$u = \cos kt \left[\frac{1}{k^2\alpha^2}\left\{\tfrac{1}{2} r^{-\frac{3}{2}} J_{i+\frac{1}{2}}(k\alpha r) - r^{-\frac{1}{2}}\frac{d}{dr} J_{i+\frac{1}{2}}(k\alpha r)\right\} Y_i\mathbf{Y}_i \right.$$
$$\left. + r^{-\frac{3}{2}} J_{i+\frac{1}{2}}(k\beta r)\, Z_i\mathbf{Z}_i\right], \tag{3}$$

$$v = \cos kt \frac{d}{d\theta}\left[-\frac{1}{k^2\alpha^2} r^{-\frac{3}{2}} J_{i+\frac{1}{2}}(k\alpha r)\, Y_i\mathbf{Y}_i + \frac{1}{i(i+1)} r^{-1}\frac{d}{dr} r^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r)\, Z_i\mathbf{Z}_i\right]$$
$$+ \cos kt \frac{1}{\sin\theta}\frac{d}{d\phi}\left\{r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r)\, W_i\mathbf{W}_i\right\}, \tag{4}$$

$$w = \cos kt \frac{1}{\sin\theta}\frac{d}{d\phi}\left[-\frac{1}{k^2\alpha^2} r^{-\frac{3}{2}} J_{i+\frac{1}{2}}(k\alpha r)\, Y_i\mathbf{Y}_i \right.$$
$$\left. + \frac{1}{i(i+1)} r^{-1}\frac{d}{dr} r^{\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r)\, Z_i\mathbf{Z}_i\right] - \cos kt \frac{d}{d\theta}\left\{r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r)\, W_i\mathbf{W}_i\right\}. \tag{5}$$

Here $J_{i+\frac{1}{2}}(x)$ denotes that solution of the Bessel equation

$$\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx} + \left\{1 - \frac{(i+\frac{1}{2})^2}{x^2}\right\}z = 0$$

which is finite, zero, at the origin $x = 0$.

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*Cambridge Philosophical Society's Transactions, vol. 15, p. 139.

†Cambridge Philosophical Society's Transactions, vol. 14, pp. 310–313.

The heavy letters, $\mathbf{Y}_i$, etc., denote surface harmonics of degree i, which may be all the same or all different, while the letters Y_i, etc., represent their arbitrary constant multipliers. The W letters answer to the "pure transverse"* vibrations in a perfect sphere; they do not occur at all in Δ. The Z letters and the Y letters with suffixes other than 0, are connected with the type "mixed radial and transverse"* vibrations. The "pure radial"* vibration may be regarded as answering to Y_0, with $\mathbf{Y}_0 = 1$.

§3. In exhibiting the values of the stresses, it is convenient to use the following abbreviations:

$$\begin{aligned} &\frac{d}{dx} J(x) = J'(x), \\ &Y_i \mathbf{Y}_i = [Y_i], \text{ etc.,} \end{aligned} \tag{6}$$

$$\left.\begin{aligned}
&r^{-\frac{1}{2}} \Big[\frac{(k\beta r)^2 - 2(i-1)(i+2)}{(k\alpha r)^2} J_{i+\frac{1}{2}}(k\alpha r) \\
&\qquad\qquad + \frac{2}{k\alpha r} \Big\{ 2J'_{i+\frac{1}{2}}(k\alpha r) - \frac{3}{k\alpha r} J_{i+\frac{1}{2}}(k\alpha r) \Big\} \Big] = {}_rA_i, \\
&r^{-\frac{5}{2}} k\beta r \Big\{ 2J'_{i+\frac{1}{2}}(k\beta r) - \frac{3}{k\beta r} J_{i+\frac{1}{2}}(k\beta r) \Big\} = {}_rB_i, \\
&- r^{-\frac{1}{2}} \frac{1}{k\alpha r} \Big\{ 2J'_{i+\frac{1}{2}}(k\alpha r) - \frac{3}{k\alpha r} J_{i+\frac{1}{2}}(k\alpha r) \Big\} = {}_rC_i, \\
&- r^{-\frac{5}{2}} \frac{k\beta r}{i(i+1)} \Big[2J'_{i+\frac{1}{2}}(k\beta r) - \frac{3}{k\beta r} J_{i+\frac{1}{2}}(k\beta r) \\
&\qquad\qquad + \Big\{ (k\beta r)^2 - 2(i-1)(1+2) \Big\} \frac{J_{i+\frac{1}{2}}(k\beta r)}{k\beta r} \Big] = {}_rD_i, \\
&r^{-\frac{1}{2}} \Big[\Big\{ \frac{(k\beta r)^2 - 2}{(k\alpha r)^2} - 2 \Big\} J_{i+\frac{1}{2}}(k\alpha r) \\
&\qquad\qquad - \frac{1}{k\alpha r} \Big\{ 2J'_{i+\frac{1}{2}}(k\alpha r) - \frac{3}{k\alpha r} J_{i+\frac{1}{2}}(k\alpha r) \Big\} \Big] = {}_rE_i, \\
&2r^{-\frac{5}{2}} J_{i+\frac{1}{2}}(k\beta r) = {}_rF_i, \\
&- \frac{2}{(k\alpha r)^2} r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\alpha r) = {}_rG_i, \\
&\frac{1}{i(i+1)} r^{-\frac{5}{2}} k\beta r \Big[2J'_{i+\frac{1}{2}}(k\beta r) - \frac{3}{k\beta r} J_{i+\frac{1}{2}}(k\beta r) + \frac{4}{k\beta r} J_{i+\frac{1}{2}}(k\beta r) \Big] = {}_rH_i.
\end{aligned}\right\} \tag{7}$$

Using these abbreviations, we have for the typical terms in the stresses employing the expressive notation of Todhunter and Pearson's "History of Elasticity . . . ."

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*See Camb. Phil. Soc. Trans., vol. 15, p. 342.

$$\widehat{rr}/n\cos kt = {}_rA_i\,[Y_i] + {}_rB_i\,[Z_i], \tag{8}$$

$$\begin{aligned}\widehat{\theta\theta}/n\cos kt = {}_rE_i\,[Y_i] + {}_rF_i\,[Z_i] &+ \frac{d^2}{d\theta^2}\{{}_rG_i\,[Y_i] + {}_rH_i\,[Z_i]\}\\ &+ \frac{d^2}{d\theta d\phi}\operatorname{cosec}\theta r\,{}_rF_i\,[W_i],\end{aligned} \tag{9}$$

$$\begin{aligned}\widehat{\phi\phi}/n\cos kt = {}_rE_i\,[Y_i] + {}_rF_i\,[Z_i] &+ \left(\frac{1}{\sin^2\theta}\,\frac{d^2}{d\phi^2} + \cot\theta\,\frac{d}{d\theta}\right)\{{}_rG_i\,[Y_i] + {}_rH_i\,[Z_i]\}\\ &- \frac{d^2}{d\theta d\phi}\operatorname{cosec}\theta r\,{}_rF_i\,[W_i],\end{aligned} \tag{10}$$

$$\widehat{r\theta}/n\cos kt = \frac{d}{d\theta}\{{}_rC_i\,[Y_i] + {}_rD_i\,[Z_i]\} + \frac{1}{\sin\theta}\,\frac{d}{d\phi}\,\tfrac{1}{2}r\,{}_rB_i\,[W_i], \tag{11}$$

$$\widehat{r\phi}/n\cos kt = \frac{1}{\sin\theta}\,\frac{d}{d\phi}\{{}_rC_i\,[Y_i] + {}_rD_i\,[Z_i]\} - \frac{d}{d\theta}\,\tfrac{1}{2}r\,{}_rB_i\,[W_i], \tag{12}$$

$$\begin{aligned}\widehat{\theta\phi}/n\cos kt = \frac{d^2}{d\theta d\phi}&\operatorname{cosec}\theta\,\{{}_rG_i\,[Y_i] + {}_rH_i\,[Z_i]\}\\ &- \left(\frac{d^2}{d\theta^2} + \tfrac{1}{2}i(i+1)\right)r\,{}_rF_i\,[W_i].\end{aligned} \tag{13}$$

§4. The surface will in every case be supposed to approach closely to that of a sphere of radius a, and we shall have frequent occasion to employ the values of the stresses when $r = a$. We shall thus for brevity write

A_i, etc., for ${}_aA_i$, etc.,
x for $k\alpha a$,
y for $k\beta a$,
$\dfrac{dA_i}{da}$, etc., for the value of $\dfrac{d}{dr}\,{}_rA_i$, etc., with r put $= a$ after differentiation.

The following relations will be found useful:

$$E_i = C_i + (y^2 - 2x^2 - 2)(A_i + 2C_i)/\{y^2 - 2(i-1)(i+2)\}, \tag{14}$$

$$F_i = -2\{B_i + i(i+1)D_i\}/\{y^2 - 2(i-1)(i+2)\}, \tag{15}$$

$$G_i = -2(A_i + 2C_i)/\{y^2 - 2(i-1)(i+2)\}, \tag{16}$$

$$H_i = \left\{\frac{y^2 - 2i(i+1)}{i(i+1)}\,B_i - 4D_i\right\}/\{y^2 - 2(i-1)(i+2)\}, \tag{17}$$

$$\begin{aligned}A_iF_i - B_iE_i = \{y^2 - 2(i-1)(i+2)\}^{-1}&[-2i(i+1)(A_iD_i - B_iC_i)\\ &- B_i\{(y^2 - 2x^2)A_i + (3y^2 - 4x^2)C_i\}],\end{aligned} \tag{18}$$

$$A_i H_i - B_i G_i = \{y^2 - 2(i-1)(i+2)\}^{-1}\left[-4(A_i D_i - B_i C_i) + \frac{y^2}{i(i+1)} A_i B_i\right], \tag{19}$$

$$a\frac{dA_i}{da} = \frac{4(y^2 - x^2)(A_i + 2C_i)}{y^2 - 2(i-1)(i+2)} - 3A_i - \tfrac{1}{2} C_i \{y^2 - 2i(i+1)\}, \tag{20}$$

$$a\frac{dB_i}{da} = -4B_i + 2\frac{y^2 - (i-1)(i+2)}{y^2 - 2(i-1)(i+2)}\{B_i + i(i+1)D_i\}, \tag{21}$$

$$a\frac{dC_i}{da} = -4C_i + 2\frac{x^2 - (i-1)(i+2)}{y^2 - 2(i-1)(i+2)}(A_i + 2C_i), \tag{22}$$

$$a\frac{dD_i}{da} = -2D_i - \frac{y^2 - 2i(i+1)}{2i(i+1)}B_i + \frac{\frac{1}{i(i+1)}B_i + D_i}{y^2 - 2(i-1)(i+2)}y^2, \tag{23}$$

$$a\frac{dF_i}{da} = B_i - F_i, \tag{24}$$

$$A_i a\frac{dB_i}{da} - B_i a\frac{dA_i}{da} = \{y^2 - 2(i-1)(i+2)\}^{-1}[2i(i+1)\{y^2 - (i-1)(i+2)\}$$
$$\times (A_i D_i - B_i C_i) - (3y^2 - 4x^2)A_i B_i + \tfrac{1}{2}\{y^4 - 4(3y^2 - 4x^2)\}B_i C_i], \tag{25}$$

$$D_i a\frac{dC_i}{da} - C_i a\frac{dD_i}{da} = \{y^2 - 2(i-1)(i+2)\}^{-1}\Big[2\{x^2 - (i-1)(i+2)\}$$
$$\times (A_i D_i - B_i C_i) - (3y^2 - 4x^2)C_i(\tfrac{1}{2}B_i + D_i) + y^2\left\{\frac{y^2 - (i-1)(i+2)}{2i(i+1)}\right\}B_i C_i\Big], \tag{26}$$

$$D_i a\frac{dA_i}{da} - C_i a\frac{dB_i}{da} = \{y^2 - 2(i-1)(i+2)\}^{-1}[-2\{y^2 - 3(i-1)(i+2)\}$$
$$\times (A_i D_i - B_i C_i) + (3y^2 - 4x^2)D_i(A_i + 2C_i) - \tfrac{1}{2}y^4 C_i D_i], \tag{27}$$

$$A_i a\frac{dD_i}{da} - B_i a\frac{dC_i}{da} = \{y^2 - 2(i-1)(i+2)\}^{-1}\Big[-\{y^2 - 4(i-1)(i+2)\}(A_i D_i - B_i C_i)$$
$$+ (3y^2 - 4x^2)B_i C_i - \frac{1}{2i(i+1)}\{y^4 - 4(i-1)(i+2)(y^2 - x^2)$$
$$- 2(3y^2 - 4x^2)\}A_i B_i\Big], \tag{28}$$

$$A_i a\frac{dD_i}{da} - B_i a\frac{dC_i}{da} + A_i F_i - B_i E_i + (i^2 + i - 3)(A_i H_i - B_i G_i)$$
$$= -\{y^2 - 2(i-1)(i+2)\}^{-1}\Big[\{y^2 + 2(i-1)(i+2)\}(A_i D_i - B_i C_i)$$
$$+ \frac{1}{2i(i+1)}y^2\{y^2 - 4(i-1)(i+2)\}A_i B_i\Big]. \tag{29}$$

The form in which several of these results are put forward is intended to facilitate their calculation when

$$A_i D_i - B_i C_i = 0.$$

§5. In any specified material x/y is to be regarded as a known numerical quantity. Perhaps the simplest material to deal with is that which, though of ordinary rigidity, is nearly incompressible; for it n is finite, but n/m, and so x/y, vanishingly small. For a vibration of finite frequency in such a material y is finite, but x very small, and in any given expression we need retain only the lowest power of x. If we do so we much simplify many of the results. Thus, for instance, putting for shortness

$$a^{-\frac{1}{2}}x^{-2}J_{i+\frac{1}{2}}(x) = X,$$

we may take

$$J'_{i+\frac{1}{2}}(x) = \tfrac{1}{2}(2i+1)\,a^{\frac{1}{2}}xX,$$

$$A_i = \{y^2 - 2i(i-1)\}\,X, \tag{30}$$

$$C_i = -2(i-1)\,X, \tag{31}$$

$$A_iD_i - B_iC_i = [\{y^2 - 2i(i-1)\}\,D_i + 2(i-1)\,B_i]\,X, \tag{32}$$

$$A_iF_i - B_iE_i = -\{y^2 - 2(i-1)(i+2)\}^{-1}\,[2i(i+1)(A_iD_i - B_iC_i) + y^2\{y^2 - 2(i-1)(i+3)\}\,B_iX], \tag{33}$$

$$A_iH_i - B_iG_i = -\{y^2 - 2(i-1)(i+2)\}^{-1}\Big[4(A_iD_i - B_iC_i) - \frac{1}{i(i+1)}\,y^2\{y^2 - 2i(i-1)\}\,B_iX\Big], \tag{34}$$

$$a\frac{dA_i}{da} = i\{y^2 - 2(i-1)(i-2)\}\,X, \tag{35}$$

$$a\frac{dC_i}{da} = -2(i-1)(i-2)\,X. \tag{36}$$

§6. In the case of free vibrations, one of the constants appearing in the expressions for the strains, stresses, etc., may be regarded as arbitrary, being determined by the amplitude of the vibration. The ratios borne to it by the other constants are to be determined from the three surface conditions:

$$\left.\begin{aligned} \lambda\,\widehat{rr} + \mu\,\widehat{r\theta} + \nu\,\widehat{r\phi} &= 0,\\ \lambda\,\widehat{r\theta} + \mu\,\widehat{\theta\theta} + \nu\,\widehat{\theta\phi} &= 0,\\ \lambda\,\widehat{r\phi} + \mu\,\widehat{\theta\phi} + \nu\,\widehat{\phi\phi} &= 0, \end{aligned}\right\} \tag{37}$$

where λ, μ, ν are the direction cosines of the outward-drawn normal relative to the fundamental directions dr, $rd\theta$, $r\sin\theta d\phi$.

If the equation to the surface be

$$r = a(1 + \epsilon\sigma), \tag{38}$$

where σ is a surface harmonic, ε a small positive or negative numerical quantity, we have as a first approximation

$$\lambda = 1\,,\ \mu = -\varepsilon\frac{d\sigma}{d\theta}\,,\quad \nu = -\varepsilon\frac{1}{\sin\theta}\,\frac{d\sigma}{d\phi}\,, \tag{39}$$

while to a second approximation

$$\left.\begin{aligned} \lambda &= 1 - \tfrac{1}{2}\varepsilon^2\left\{\left(\frac{d\sigma}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\,\frac{d\sigma}{d\phi}\right)^2\right\}, \\ \mu &= -\varepsilon\,(1 - \varepsilon\sigma)\frac{d\sigma}{d\theta}\,, \\ \nu &= -\varepsilon\,(1 - \varepsilon\sigma)\frac{1}{\sin\theta}\,\frac{d\sigma}{d\phi}\,. \end{aligned}\right\} \tag{40}$$

SECTION II.

§7. The first application of the method is to the case of approximately pure radial vibrations in the material contained by the surface

$$r = a\,(1 + \varepsilon_i\sigma_i)\,, \tag{1}$$

σ_i being a surface harmonic of degree i, and ε_i a small numerical quantity.

If the surface were truly spherical, the displacement

$$u = \cos kt\,Y_0\,\frac{r^{-\frac{1}{2}}}{ka}\left\{\tfrac{1}{2}\,\frac{J_{\frac{1}{2}}(kar)}{kar} - J'_{\frac{1}{2}}(kar)\right\}, \tag{2}$$

where Y_0 is an arbitrary constant, would alone exist. The frequency $k/2\pi$ of the fundamental and higher vibrations would be given by

$$\widehat{(rr)}_{r=a} = 0\,,$$

or
$$_aA_0 = 0\,. \tag{3}$$

Let us now assume that when (1) is the equation to the surface there is a type of vibration in which (2) is the *principal* term, there being other necessary *subsidiary* terms of the order ε_i of small quantities.

Supposing we go as far as terms in ε_i^2, then the contribution of (2) to the first of the surface conditions (37), Section I, is, omitting the $\cos kt$,

$$\left[1 + \varepsilon_i\sigma_i\,a\frac{d}{da} + \tfrac{1}{2}(\varepsilon_i\sigma_i)^2a^2\frac{d^2}{da^2} - \tfrac{1}{2}\varepsilon_i^2\left\{\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\,\frac{d\sigma_i}{d\phi}\right)^2\right\}\right]A_0Y_0\,.$$

In this Y_0 is treated as a constant, A_0 as a function of a, the abbreviation A_i for $_aA_i$, etc., being used as in Section I.

Similarly, omitting likewise $\cos kt$, the contributions of (2) to the second and third surface conditions are respectively

$$-\varepsilon_i\frac{d\sigma_i}{d\theta}\left(1-\varepsilon_i\sigma_i+\varepsilon_i\sigma_i a\frac{d}{da}\right)E_0Y_0,\text{ and }-\varepsilon_i\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\left(1-\varepsilon_i\sigma_i+\varepsilon_i\sigma_i a\frac{d}{da}\right)E_0Y_0.$$

We can get rid of all the terms of order ε_i in the surface conditions by introducing as *subsidiary* terms those typical terms in (3), (4), (5), Section I, whose coefficients are Y_i and Z_i, putting

$$\mathbf{Y}_i=\mathbf{Z}_i=\sigma_i.$$

To get rid of terms of order ε_i^2 other harmonic terms would be required, but as the knowledge of these is not actually required for our present purpose, we need only mention their existence.

The first surface condition may now be written

$$\begin{aligned}&\left[1+\varepsilon_i\sigma_i a\frac{d}{da}+\tfrac{1}{2}(\varepsilon_i\sigma_i)^2a^2\frac{d^2}{da^2}-\tfrac{1}{2}\varepsilon_i^2\left\{\left(\frac{d\sigma_i}{d\theta}\right)^2+\left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\}\right]A_0Y_0\\&\quad+\left(1+\varepsilon_i\sigma_i a\frac{d}{da}\right)(A_iY_i+B_iZ_i)\sigma_i-\varepsilon_i(C_iY_i+D_iZ_i)\left\{\left(\frac{d\sigma_i}{d\theta}\right)^2+\left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\}\\&\quad+\text{ other harmonic terms of order }\varepsilon_i^2=0.\end{aligned}\tag{4}$$

For the second surface condition we have

$$\begin{aligned}&-\varepsilon_i\frac{d\sigma_i}{d\theta}\left(1-\varepsilon_i\sigma_i+\varepsilon_i\sigma_i a\frac{d}{da}\right)E_0Y_0+\left(1+\varepsilon_i\sigma_i a\frac{d}{da}\right)\frac{d\sigma_i}{d\theta}(C_iY_i+D_iZ_i)\\&\quad-\varepsilon_i\frac{d\sigma_i}{d\theta}\sigma_i(E_iY_i+F_iZ_i)-\tfrac{1}{2}(G_iY_i+H_iZ_i)\frac{d}{d\theta}\left\{\left(\frac{d\sigma_i}{d\theta}\right)^2+\left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\}\\&\quad+\text{ other harmonics of order }\varepsilon_i^2=0.\end{aligned}\tag{5}$$

The third surface condition may be derived from the second by writing $\frac{1}{\sin\theta}\frac{d}{d\phi}$ for $\frac{d}{d\theta}$.

In all cases Y_i and Z_i are to be treated as constants during differentiations.

§8. The coefficient of σ_i in (4) must vanish, thus we have as a first approximation

$$A_iY_i+B_iZ_i=-Y_0\varepsilon_i a\frac{dA_0}{da}.\tag{6}$$

Again the coefficient of $\frac{d\sigma_i}{d\theta}$ in (5)—which is also the coefficient of $\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}$ in the third surface condition—must vanish; thus to a first approximation

$$C_i Y_i + D_i Z_i = Y_0 \varepsilon_i E_0. \tag{7}$$

From (6) and (7) we get as first approximations

$$\left.\begin{aligned} Y_i &= -\varepsilon_i Y_0 \left(D_i a \frac{dA_0}{da} + B_i E_0\right) \div (A_i D_i - B_i C_i), \\ Z_i &= \varepsilon_i Y_0 \left(C_i a \frac{dA_0}{da} + A_i E_0\right) \div (A_i D_i - B_i C_i). \end{aligned}\right\} \tag{8}$$

Returning to (4) we obtain the frequency equation by equating to zero the terms independent of surface harmonics. Since in the constant terms the coefficients of Y_i and Z_i are of order ε_i, it suffices to substitute for these the first approximations (8); before this substitution is effected, the differentiations indicated in (4) have to be carried out.

Since the first approximation to the frequency equation is

$$A_0 = 0,$$

it is obvious that the terms arising from

$$-\tfrac{1}{2}\varepsilon_i^2 \left\{\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\} A_0 Y_0$$

may be neglected.

Thus when powers of ε_i above the second are neglected, we get for the frequency equation, dividing out by Y_0,

$$\begin{aligned} A_0 + \varepsilon_i^2 \Bigg[& \left\{ \tfrac{1}{2} a^2 \frac{d^2 A_0}{da^2} + \frac{a \frac{dB_i}{da}\left(C_i a \frac{dA_0}{da} + A_i E_0\right) - a\frac{dA_i}{da}\left(D_i a \frac{dA_0}{da} + B_i E_0\right)}{A_i D_i - B_i C_i} \right\} \\ & \times \left\{\text{const. term in } \sigma_i^2\right\} - E_0 \left\{\text{const. term in} \left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\}\Bigg] = 0. \end{aligned} \tag{9}$$

Supposing the effect of the departure of the surface from the truly spherical form has been to increase the frequency by the amount $\delta k/2\pi$, then $k - \delta k$ must be a solution of the equation

$$A_0 = 0.$$

Consequently, neglecting δk^2, we must have

$$A_0 - \frac{\delta k}{k} k \frac{dA_0}{dk} = 0.$$

But $a^{\frac{1}{2}} A_0$ being a function of ka, and A_0 being to a first approximation zero, we may in the coefficient of $\delta k / k$ put

$$k \frac{dA_0}{dk} = a \frac{dA_0}{da},$$

and thus get

$$A_0 - \frac{\delta k}{k} a \frac{dA_0}{da} = 0. \tag{10}$$

The two frequency equations (9) and (10) must be identical, thus

$$\frac{\delta k}{k} = \frac{-\varepsilon_i^2}{a \dfrac{dA_0}{da}} \left[\left\{ \tfrac{1}{2} a^2 \frac{d^2 A_0}{da^2} + \frac{a \dfrac{dB_i}{da}\left(C_i a \dfrac{dA_0}{da} + A_i E_0\right) - a \dfrac{dA_i}{da}\left(D_i a \dfrac{dA_0}{da} + B_i E_0\right)}{A_i D_i - B_i C_i} \right\} \right.$$
$$\left. \times \left\{ \text{const. term in } \sigma_i^2 \right\} - E_0 \left\{ \text{const. term in } \left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta} \frac{d\sigma_i}{d\phi}\right)^2 \right\} \right] \tag{11}$$

§9. It is a simple matter to find the constant terms in σ_i^2 and

$$\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta} \frac{d\sigma_i}{d\phi}\right)^2 \text{ expanded in surface harmonics.}$$

It is in fact obvious that, putting $\cos\theta = \mu$,

$$\int_{-1}^{+1} \int_0^{2\pi} \sigma_i^2 \, d\mu \, d\phi = \int_{-1}^{+1} \int_0^{2\pi} (\text{constant term in } \sigma_i^2) \, d\mu \, d\phi ;$$

$$\therefore \text{ constant term in } \sigma_i^2 = \frac{1}{4\pi} \int_{-1}^{+1} \int_0^{2\pi} \sigma_i^2 \, d\mu \, d\phi. \tag{12}$$

The value of this integral is known for the ordinary forms of Laplace Functions. For instance,* if

$$\sigma_i = P_{i,\kappa}(\mu) \cos \kappa\phi = \frac{\underline{|i - \kappa}}{1 . 3 \ldots . (2i - 1)} (1 - \mu^2)^{\frac{\kappa}{2}} \frac{d^\kappa}{d\mu^\kappa} P_i(\mu) \cos \kappa\phi,$$

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* See Ferrers' "Spherical Harmonics," pp. 85, 86, or Todhunter's "Functions of Laplace . . . . ," p. 158.

where P_i denotes i^{th} zonal harmonic and $\varkappa$ is integral,

$$\int_{-1}^{+1}\int_{0}^{2\pi}\sigma_i^2 d\mu\, d\phi = 2\pi\, \underline{|i-\varkappa}\, \underline{|i+\varkappa} \div \left[(2i+1)\{1.3\ldots.(2i-1)\}^2\right].$$

Similarly constant term in $\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2$

$$= \frac{1}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi}\left\{\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2\right\} d\mu\, d\phi$$

$$= \frac{i(i+1)}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi}\sigma_i^2 d\mu\, d\phi$$

(see Cambridge Philosophical Society's Transactions, vol. 16, p. 33).

Thus constant term in $\left(\frac{d\sigma_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{d\sigma_i}{d\phi}\right)^2$

$$= i(i+1)(\text{constant term in } \sigma_i^2). \tag{13}$$

Employing (12) and (13), we throw (11) into the form

$$\frac{\delta k}{k} = -\frac{\epsilon_i^2}{a\frac{dA_0}{da}}\,\frac{1}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi}\sigma_i^2\, d\mu\, d\phi \times \left[\tfrac{1}{2}a^2\frac{d^2A_0}{da^2} - i(i+1)E_0 \right.$$
$$\left. + \frac{a\frac{dA_0}{da}\left(C_i a\frac{dB_i}{da} - D_i a\frac{dA_i}{da}\right) + E_0\left(A_i a\frac{dB_i}{da} - B_i a\frac{dA_i}{da}\right)}{A_iD_i - B_iC_i}\right]. \tag{14}$$

§10. For a given value of i, the change of pitch is the same in all cases in which $\int_{-1}^{+1}\int_{0}^{2\pi}\sigma_i^2 d\mu\, d\phi$ has a constant value; its sign is independent of that of ϵ_i.

We may simplify (14) by means of the relation

$$\frac{\epsilon_i^2}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi}\sigma_i^2 d\mu\, d\phi = \frac{2(\delta S_i/S)}{i^2+i+2}, \tag{15}$$

where S is the area of the surface of a sphere of radius a, $S+\delta S_i$ that of the surface (1).

We may also employ the first approximation frequency equation (3) inside the square bracket; doing so, we may put

$$\left.\begin{aligned} a^2\frac{d^2A_0}{da^2}\Big/ a\frac{dA_0}{da} &= -2\frac{(kaa)^2(m+n)^2-8n(3m-n)}{(kaa)^2(m+n)^2-4n(3m-n)} = -2\frac{y^4-8(3y^2-4x^2)}{y^4-4(3y^2-4x^2)}, \\ E_0\Big/ a\frac{dA_0}{da} &= -\frac{2n(3m-n)}{(kaa)^2(m+n)^2-4n(3m-n)} = \frac{-2(3y^2-4x^2)}{y^4-4(3y^2-4x^2)}, \end{aligned}\right\} \qquad (16)$$

where x and y have their previous significations.

We may thus replace (14) by

$$\begin{aligned} \frac{\delta k}{k} = \frac{2(\delta S_i/S)}{i^2+i+2}\{y^4-4(3y^2-4x^2)\}^{-1}\Big[&y^4-2(i^2+i+4)(3y^2-4x^2) \\ &+\{y^4-4(3y^2-4x^2)\}\left(D_ia\frac{dA_i}{da}-C_ia\frac{dB_i}{da}\right)\Big/(A_iD_i-B_iC_i) \\ &+2(3y^2-4x^2)\left(A_ia\frac{dB_i}{da}-B_ia\frac{dA_i}{da}\right)\Big/(A_iD_i-B_iC_i)\Big]. \end{aligned} \qquad (17)$$

The employment of the results (25) and (27) of Section I renders it unnecessary to actually perform the differentiations occurring on the right of (17). The values to be assigned to x and y in (17) being determined by (3), are to be regarded as known quantities. Professor Lamb* has actually determined the values of x/π for the six notes of lowest pitch for a series of values of Poisson's ratio.

SECTION III.

§11. In a perfect sphere the vibrations next in simplicity to the pure radial are the pure transverse. We thus proceed to consider some vibrations approximately of this type in nearly spherical solids.

The first case we shall consider is when the surface is the nearly spherical spheroid

$$r = a(1+\epsilon_2 P_2), \qquad (1)$$

and the principal term in the displacement is

$$w = -W_i \cos kt \frac{dP_i}{d\theta} r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r), \qquad (2)$$

where W_i is a constant, and the rest of the notation is as before.

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* Proceedings London Mathematical Society, vol. 13, p. 202; see also Trans. Camb. Phil. Soc., vol. 15, p. 157, or Love's "Treatise on Elasticity," vol. 1, p. 318.

The stresses $\widehat{r\phi}$, $\widehat{\theta\phi}$ answering to (2) would be given over a spherical surface of radius a by

$$\left.\begin{aligned} \widehat{r\phi}/n \cos kt &= -\tfrac{1}{2} W_i a\, B_i \frac{dP_i}{d\theta}, \\ \widehat{\theta\phi}/n \cos kt &= -\tfrac{1}{2} W_i a\, F_i \left\{ i(i+1) P_i + 2\frac{d^2 P_i}{d\theta^2} \right\}, \end{aligned}\right\} \tag{3}$$

where B_i stands for ${}_aB_i$, etc.

In the present instance the change in the frequency has a term of the order ε_2, so we may neglect terms of order ε_2^2 and take for the surface condition

$$\widehat{r\phi} - \varepsilon_2 \frac{dP_2}{d\theta} \widehat{\theta\phi} = 0. \tag{4}$$

Having regard to (4) and to the relations

$$P_2 \frac{dP_i}{d\theta} = \frac{3i(i+1)}{2(2i-1)(2i+1)} \frac{dP_{i-2}}{d\theta} + \frac{i^2+i-3}{(2i-1)(2i+3)} \frac{dP_i}{d\theta} + \frac{3i(i+1)}{2(2i+1)(2i+3)} \frac{dP_{i+2}}{d\theta}, \tag{5}$$

$$\frac{dP_2}{d\theta} \left\{ i(i+1) P_i + 2\frac{d^2P_i}{d\theta^2} \right\} = \frac{3i(i+1)(i+2)}{(2i-1)(2i+1)} \frac{dP_{i-2}}{d\theta} + \frac{9(i-1)(i+2)}{(2i-1)(2i+3)} \frac{dP_i}{d\theta} - \frac{3(i-1)i(i+1)}{(2i+1)(2i+3)} \frac{dP_{i+2}}{d\theta}, \tag{6}$$

we see that the addition to (2) of subsidiary terms of the same type in $\dfrac{dP_{i-2}}{d\theta}$ and $\dfrac{dP_{i+2}}{d\theta}$ is sufficient to solve the problem. We thus assume in place of (2)

$$w = -\cos kt \frac{d}{d\theta} \Big[W_{i-2} P_{i-2} r^{-\frac{1}{2}} J_{i-\frac{3}{2}}(k\beta r) + W_i P_i r^{-\frac{1}{2}} J_{i+\frac{1}{2}}(k\beta r) + W_{i+2} P_{i+2} r^{-\frac{1}{2}} J_{i+\frac{5}{2}}(k\beta r) \Big], \tag{7}$$

where we know W_{i-2}, W_{i+2} to be of the order ε_2 of small quantities.

Referring to (3), (5), (6) and (7), we see that the surface condition leads to

$$\begin{aligned} W_i \Big[aB_i \frac{dP_i}{d\theta} &+ \varepsilon_2 a \frac{d}{da}(aB_i) \frac{d}{d\theta} \left\{ \frac{3i(i+1)}{2(2i-1)(2i+1)} P_{i-2} \right. \\ &\left. + \frac{i^2+i-3}{(2i-1)(2i+3)} P_i + \frac{3i(i+1) P_{i+2}}{2(2i+1)(2i+3)} \right\} \\ &+ \varepsilon_2 a F_i \frac{d}{d\theta} \left\{ -\frac{3i(i+1)(i+2)}{(2i-1)(2i+1)} P_{i-2} - \frac{9(i-1)(i+2)}{(2i-1)(2i+3)} P_i \right. \\ &\left. + \frac{3(i-1)i(i+1)}{(2i+1)(2i+3)} P_{i+2} \right\} \Big] \\ &+ W_{i-2} aB_{i-2} \frac{dP_{i-2}}{d\theta} + W_{i+2} aB_{i+2} \frac{dP_{i+2}}{d\theta} = 0. \end{aligned} \tag{8}$$

As this holds all over the surface, we may equate separately to zero the coefficients of the differentials of the three harmonics. Doing so, we find

$$W_{i-2}/W_i = \frac{3i(i+1)\varepsilon_2}{(2i-1)(2i+1)}\left\{(i+2)\,aF_i - \tfrac{1}{2}a\frac{d}{da}(aB_i)\right\}\Big/ aB_{i-2}. \tag{9}$$

$$aB_i + \frac{\varepsilon_2}{(2i-1)(2i+3)}\left\{(i^2+i-3)\,a\frac{d}{da}(aB_i) - 9(i-1)(i+2)\,aF_i\right\} = 0, \tag{10}$$

$$W_{i+2}/W_i = -\frac{3i(i+1)\varepsilon_2}{(2i+1)(2i+3)}\left\{(i-1)\,aF_i + \tfrac{1}{7}a\frac{d}{da}(aB_i)\right\}\Big/ aB_{i+2}. \tag{11}$$

The value of W_i is supposed to be given by the known amplitude of vibration, so that (9) and (11) determine W_{i-2} and W_{i+2}.

§12. For the frequency equation we have (10), which becomes for the perfect sphere

$$B_i = 0. \tag{12}$$

Reasoning similar to that in Section II leads us to the conclusion that if $\delta k/2\pi$ be the increment in the pitch due to the departure from the spherical form, the equation

$$aB_i - \frac{\delta k}{k}\,a\frac{d}{da}(aB_i) = 0$$

must be identical with (10).

Thus we get

$$\frac{\delta k}{k} = -\frac{\varepsilon_2}{(2i-1)(2i+3)}\left\{i^2+i-3-9(i-1)(i+2)\frac{aF_i}{a\dfrac{d}{da}(aB_i)}\right\}. \tag{13}$$

In the coefficient of ε_2 we may regard k as possessing the values it would have in a perfect sphere, i. e. as being a root of (12); and it is easily shown from (15) and (21), Section I, that when $B_i = 0$,

$$a\frac{dB_i}{da} = -F_i\{y^2-(i-1)(i+2)\}. \tag{14}$$

Employing this result in (13) we get

$$\frac{\delta k}{k} = -\varepsilon_2\frac{(i^2+i-3)\,y^2-(i-3)(i-1)(i+2)(i+4)}{(2i-1)(2i+3)\{y^2-(i-1)(i+2)\}}, \tag{15}$$

where $y \equiv k\beta a$ is the root of

$$2yJ'_{i+\frac{1}{2}}(y) - 3J_{i+\frac{1}{2}}(y) = 0, \tag{16}$$

which answers to the note of the transverse type under consideration.

§13. The functions $J_{i+\frac{1}{2}}(y)$ take conveniently concise forms involving sines and cosines. These will be found for values of i up to 5 on p. 42 of Gray and Mathews' "Treatise on Bessel Functions." Their substitution will be found to lead to comparatively simple frequency equations for small values of i. The equations answering to $i = 1$ and $i = 2$ were discussed by Professor Lamb in his valuable paper in the Proc. of the London Math. Soc., vol. 13 (see his equations (38) and (43)), and the numerical values of the smaller roots will be found there and in Love's "Treatise on .... Elasticity," vol. 1, pp. 318–9.

In the case $i = 1$ we need not trouble about the form or roots of (16) for our present purpose, because (15) reduces to the simple form

$$\delta k/k = \epsilon_2/5. \tag{17}$$

The pitch is thus altered in the same proportion in all the notes; it is raised or lowered according as the surface is prolate or oblate.

For $i = 2$, (16) takes the form

$$\tan y = y\,(y^2 - 12)/(5y^2 - 12), \tag{18}$$

and the six lowest notes are approximately given by

$$y/\pi = .7961,\ 2.2715,\ 3.3469,\ 4.3837,\ 5.4059,\ 6.4209.$$

In this case (15) becomes

$$\frac{\delta k}{k} = -\frac{\epsilon_2}{7}\,\frac{y^2 + 8}{y^2 - 4}. \tag{19}$$

It follows from the above values of the roots that δk is here always opposite in sign to ϵ_2, or the pitch is raised or lowered according as the spheroid is oblate or prolate. For the higher notes, when $i = 2$,

$$\delta k/k = -\epsilon_2/7 \tag{20}$$

becomes an increasingly close approximation.

For all values of i successive higher roots of (16) become approximately successive even or successive odd multiples of $\pi/2$. Thus when i is not large we get from (15) as an approximation for all the higher "harmonics"

$$\delta k/k = -\varepsilon_2(i^2+i-3)/\{(2i-1)(2i+3)\}, \tag{21}$$

a result which, of course, includes (20).

On the other hand, when i is very large we get as an approximation in the case of the fundamental note and lower harmonics

$$\delta k/k = -\varepsilon_2/4. \tag{22}$$

This last result, and likewise (21) when i exceeds 1, represent a change of pitch opposite in sign to ε_2.

SECTION IV.

§14. As a second example of transverse vibrations, consider the solid contained by the approximately spherical surface

$$r = a(1+\varepsilon_i P_i) \tag{1}$$

vibrating approximately in the *rotatory** type given by

$$w/\cos kt = w_1 \sin\theta\, r^{-\frac{1}{2}} J_{\frac{3}{2}}(k\beta r), \tag{2}$$

where w_1 is a constant.

Answering to a displacement

$$w/\cos kt = -W_j\frac{dP_j}{d\theta}\, r^{-\frac{1}{2}} J_{j+\frac{1}{2}}(k\beta r), \tag{3}$$

where W_j is a constant, we have over $r=a$,

$$\widehat{r\phi}/n \cos kt = -\tfrac{1}{2} W_j a B_j \frac{dP_j}{d\theta}, \tag{4}$$

$$\widehat{\theta\phi}/n \cos kt = \tfrac{1}{2} W_j a F_j \left\{ j(j+1)P_j - 2\mu\frac{dP_j}{d\mu}\right\}, \tag{5}$$

where $\mu \equiv \cos\theta$.

For $j=1$, $\widehat{\theta\phi}$ vanishes and so the contribution of (2) to the surface condition

$$\widehat{r\phi} - \varepsilon_i \frac{dP_i}{d\theta}\,\widehat{\theta\phi} = 0 \tag{6}$$

is only

$$\left(1+\varepsilon_i P_i a\frac{d}{da}\right) W_1 \sin\theta\, a B_1/2,$$

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*Professor Lamb's term for case $i=1$.

i. e.*
$$\left\{1 + \frac{\varepsilon_i}{2i+1} \frac{d}{d\mu} (P_{i+1} - P_{i-1})\, a \frac{d}{da}\right\} W_1 \sin\theta\, a B_1/2 .$$

It is thus obvious that when we neglect terms in ε_i^2 it suffices to add to the value of $w/\cos kt$ given in (2) the subsidiary terms

$$-\frac{d}{d\theta}\left[W_{i-1} P_{i-1} r^{-\frac{1}{2}} J_{i-\frac{1}{2}}(k\beta r) + W_{i+1} P_{i+1} r^{-\frac{1}{2}} J_{i+\frac{3}{2}}(k\beta r)\right],$$

where W_{i-1}, W_{i+1} are constants.

From principal and subsidiary terms combined we get as the surface condition

$$\begin{aligned} aB_1 + \varepsilon_i a \frac{d}{da}(aB_1) \frac{1}{2i+1} \frac{d}{d\mu}(P_{i+1} - P_{i-1}) \\ + (W_{i-1}/W_1) aB_{i-1} \frac{dP_{i-1}}{d\mu} + (W_{i+1}/W_1) aB_{i+1} \frac{dP_{i+1}}{d\mu} . \end{aligned} \quad (7)$$

As this holds all over the surface, we may equate separately to zero the coefficients of $\frac{dP_{i-1}}{d\mu}$, $\frac{dP_{i+1}}{d\mu}$ and the constant term. We thus find

$$W_{i-1} = \frac{1}{2i+1} W_1 \varepsilon_i \frac{\frac{d}{da}(aB_1)}{B_{i-1}}, \quad (8)$$

$$W_{i+1} = -\frac{1}{2i+1} W_1 \varepsilon_i \frac{\frac{d}{da}(aB_1)}{B_{i+1}}, \quad (9)$$

and finally for the frequency equation,

$$aB_1 = 0. \quad (10)$$

Since (10) is identical with the frequency equation in the perfect sphere, $r = a$, the change of frequency in the present case is zero when terms of order ε_i^2 are neglected.

This at first sight appears inconsistent with the result obtained in (17) of Section III, that in the special case of the surface

$$r = a(1 + \varepsilon_2 P_2)$$

there is, in the case of the vibrations of approximately rotatory type, a change of pitch given by

$$\delta k/k = \tfrac{1}{5}\varepsilon_2 .$$

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* See equation (11), p. 46 of Todhunter's "Functions of Laplace . . . ."

The explanation is simply that $i = 2$ constitutes an exceptional case. The work of the present section in reality tacitly assumes that $\frac{dP_1}{d\mu}$, $\frac{dP_{i-1}}{d\mu}$ and $\frac{dP_{i+1}}{d\mu}$ are all different, which is not true when $i = 2$. The case $i = 2$ must thus be excluded from the range of the present section; it clearly is the only case in which the tacit assumption is not true.

§15. When we desire to go as far as terms in ε_i^2, we get in place of (7)

$$\begin{aligned} & W_1 \left\{ aB_1 + \tfrac{1}{2}\varepsilon_i^2 a^2 \frac{d^2}{da^2}(aB_1) \times \text{constant term in } P_i^2 \right\} \\ & \quad + \text{constant terms in} \Big[\varepsilon_i P_i a \frac{d}{da} \left\{ W_{i-1} a B_{i-1} \frac{dP_{i-1}}{d\mu} + W_{i+1} a B_{i+1} \frac{dP_{i+1}}{d\mu} \right\} \\ & \quad + \varepsilon_i \frac{dP_i}{d\mu} W_{i-1} a F_{i-1} \left\{ (i-1)\, i P_{i-1} - 2\mu \frac{dP_{i-1}}{d\mu} \right\} \\ & \quad + \varepsilon_i \frac{dP_i}{d\mu} W_{i+1} a F_{i+1} \left\{ (i+1)(i+2) P_{i+1} - 2\mu \frac{dP_{i+1}}{d\mu} \right\} \Big] = 0. \end{aligned} \tag{11}$$

By the "constant term" in P_i^2, for instance, is here meant the value of A_1 in the series

$$P_i^2 = A_1 + A_3 \frac{dP_3}{d\mu} + \ldots\ldots + A_{2i+1} \frac{dP_{2i+1}}{d\mu},$$

or

$$\begin{aligned} P_i^2 &= (A_1 + A_3 + \ldots\ldots A_{2i+1}) + 5P_2 (A_3 + \ldots\ldots + A_{2i+1}) + \ldots\ldots \\ &= C_0 + C_2 P_2 + \ldots\ldots, \text{ say.} \end{aligned}$$

To determine A_1 we notice that

$$C_0 = \tfrac{1}{2} \int_{-1}^{+1} P_i^2 \, d\mu = \frac{1}{2i+1}. \tag{12}$$

Again,

$$C_2 \int_{-1}^{+1} P_2^2 \, d\mu = \int_{-1}^{+1} P_2 P_i^2 \, d\mu. \tag{13}$$

Now it is easy to prove

$$P_2 P_i = \frac{3(i-1)i}{2(2i-1)(2i+1)} P_{i-2} + \frac{i(i+1)}{(2i-1)(2i+3)} P_i + \frac{3(i+1)(i+2)}{2(2i+1)(2i+3)} P_{i+2};$$

therefore

$$\begin{aligned} \int_{-1}^{+1} P_2 P_i^2 \, d\mu &= \frac{i(i+1)}{(2i-1)(2i+3)} \int_{-1}^{+1} P_i^2 \, d\mu \\ &= 2i(i+1) \div \{(2i-1)(2i+1)(2i+3)\}. \end{aligned}$$

Hence by (13)

$$C_2 = 5i(i+1) \div \{(2i-1)(2i+1)(2i+3)\}. \tag{14}$$

But by the equivalence of the expressions for P_i^2,

$$A_1 = C_0 - \tfrac{1}{5}C_2,$$

$$\therefore\ A_1 = 3(i^2+i-1) \div \{(2i-1)(2i+1)(2i+3)\}. \tag{15}$$

In a somewhat similar fashion we find, with like forms of expansion in *differentials* of harmonics,

$$\text{constant term in } P_i \frac{dP_{i-1}}{d\mu} = -3(i-1)i \div \{2(2i-1)(2i+1)\}, \tag{16}$$

$$\text{constant term in } P_i \frac{dP_{i+1}}{d\mu} = 3(i+1)(i+2) \div \{2(2i+1)(2i+3)\}, \tag{17}$$

$$\text{constant term in } \frac{dP_i}{d\mu}\left\{i(i-1)P_{i-1} - 2\mu\frac{dP_{i-1}}{d\mu}\right\} = \frac{3(i-2)(i-1)i(i+1)}{2(2i-1)(2i+1)}, \tag{18}$$

$$\text{constant term in } \frac{dP_i}{d\mu}\left\{(i+1)(i+2)P_{i+1} - 2\mu\frac{dP_{i+1}}{d\mu}\right\}$$

$$= -\frac{3i(i+1)(i+2)(i+3)}{2(2i+1)(2i+3)}. \tag{19}$$

We thus convert (11) into

$$\begin{aligned}
aB_1 &+ \tfrac{3}{2}\epsilon_i^2 \frac{i^2+i-1}{(2i-1)(2i+1)(2i+3)} a^2 \frac{d^2}{da^2}(aB_1) \\
&- \epsilon_i (W_{i-1}/W_1) \frac{3(i-1)i}{2(2i-1)(2i+1)} \left\{ a\frac{d}{da}(aB_{i-1}) - (i-2)(i+1)aF_{i-1} \right\} \\
&+ \epsilon_i (W_{i+1}/W_1) \frac{3(i+1)(i+2)}{2(2i+1)(2i+3)} \left\{ a\frac{d}{da}(aB_{i+1}) - i(i+3)aF_{i+1} \right\} = 0.
\end{aligned}$$

Substituting from (8) and (9), we find

$$\begin{aligned}
aB_1 - \frac{3\epsilon_i^2}{2(2i+1)} \Bigg[&- \frac{i^2+i-1}{(2i-1)(2i+3)} a^2 \frac{d^2}{da^2}(aB_1) \\
&+ \frac{(i-1)i}{(2i-1)(2i+1)} \frac{a\frac{d}{da}(aB_1)}{aB_{i-1}} \left\{ a\frac{d}{da}(aB_{i-1}) - (i-2)(i+1)aF_{i-1} \right\} \\
&+ \frac{(i+1)(i+2)}{(2i+1)(2i+3)} \frac{a\frac{d}{da}(aB_1)}{aB_{i+1}} \left\{ a\frac{d}{da}(aB_{i+1}) - i(i+3)aF_{i+1} \right\} \Bigg] = 0.
\end{aligned}$$

§16. Reasoning as before from the identity of this with

$$aB_1 - \frac{\delta k}{k} a \frac{d}{da}(aB_1) = 0,$$

we find for the increment in pitch due to the departure from the truly spherical form

$$\frac{\delta k}{k} = \tfrac{3}{2} \frac{\epsilon_i^2}{2i+1} \left[- \frac{i^2+i-1}{(2i-1)(2i+3)} \frac{a^2 \dfrac{d^2}{da^2}(aB_1)}{a \dfrac{d}{da}(aB_1)} \right.$$
$$+ \frac{(i-1)i}{(2i-1)(2i+1)} \frac{a \dfrac{d}{da}(aB_{i-1}) - (i-2)(i+1) aF_{i-1}}{aB_{i-1}}$$
$$\left. + \frac{(i+1)(i+2)}{(2i+1)(2i+3)} \frac{a \dfrac{d}{da}(aB_{i+1}) - i(i+3) aF_{i+1}}{aB_{i+1}} \right]. \qquad (20)$$

In the coefficient of ϵ_i^2 we substitute for k, after differentiation, that root of (10) which gives the first approximation to the frequency of the vibration under consideration. For any such value of k it is easy to prove

$$a^2 \frac{d^2}{da^2}(aB_1) \div a \frac{d}{da}(aB_1) = -2.$$

Also in general from (15), (21), etc., Section I,

$$\frac{a \dfrac{d}{da}(aB_j) - (j-1)(j+2) aF_j}{aB_j} = -3 - y^2 F_j / B_j.$$

Thus we can replace (20) by the more concise formula

$$\frac{\delta k}{k} = -\tfrac{3}{2} \frac{\epsilon_i^2}{2i+1} \left[\frac{4(i^2+i-1)}{(2i-1)(2i+3)} \right.$$
$$\left. + \frac{y^2}{2i+1} \left\{ \frac{(i-1) i F_{i-1}}{(2i-1) B_{i-1}} + \frac{(i+1)(i+2) F_{i+1}}{(2i+3) B_{i+1}} \right\} \right]. \qquad (21)$$

This can be further simplified by the result

$$B_s / F_s = \{ y J'_{s+\frac{1}{2}}(y) \} / J_{s+\frac{1}{2}}(y) - \tfrac{3}{2}, \qquad (22)$$

deducible at once from (7), Section I.

The values to be ascribed to y are the roots of (10), whose approximate values* are given by

$$y/\pi = 1{\cdot}8346,\ 2{\cdot}8950,\ 3{\cdot}9225,\ 4{\cdot}9385,\ 5{\cdot}9489,\ 6{\cdot}9563, \text{ etc.}$$

The algebraical expressions for the Bessels involved for value of s from 1 to 5 are given, as already stated, on p. 42 of Gray and Mathews' Treatise, but in the absence of a complete table of numerical values, the numerical evaluation of $\delta k/k$ from (21) would prove somewhat tedious.

SECTION V.

§17. The third and most complicated type of vibration in the perfect sphere is the "mixed radial and transverse." As an example in which the principal terms are of this type, we may consider the vibrations in the nearly spherical spheroid

$$r = a(1 + \varepsilon_2 P_2), \tag{1}$$

for which the principal terms in the displacements are those with the coefficients Y_i, Z_i in (3), (4) and (5), Section I. We shall take the simplest case when

$$\mathbf{Y}_i = \mathbf{Z}_i = P_i.$$

The requisite subsidiary terms are also of the mixed radial and transverse types, and depend on the harmonics P_{i-2} and P_{i+2}.

When terms of order ε_2^2 are neglected, it will be seen from (8) to (13) and (37) of Section I that the first surface equation is

$$\begin{aligned}\left(1 + \varepsilon_2 P_2 a \frac{d}{da}\right)[P_i(A_iY_i + B_iZ_i)] - \varepsilon_2 \frac{dP_2}{d\theta}\frac{dP_i}{d\theta}(C_iY_i + D_iZ_i)\\ + P_{i-2}(A_{i-2}Y_{i-2} + B_{i-2}Z_{i-2}) + P_{i+2}(A_{i+2}Y_{i+2} + B_{i+2}Z_{i+2}) = 0.\end{aligned} \tag{2}$$

In like manner the second surface condition is

$$\begin{aligned}\left(1 + \varepsilon_2 P_2 a \frac{d}{da}\right)\frac{dP_i}{d\theta}(C_iY_i + D_iZ_i)\\ - \varepsilon_2 \frac{dP_2}{d\theta}\left\{P\,(E_iY_i + F_iZ_i) + \frac{d^2P_i}{d\theta^2}(G_iY_i + H_iZ_i)\right\}\\ + \frac{dP_{i-2}}{d\theta}(C_{i-2}Y_{i-2} + D_{i-2}Z_{i-2}) + \frac{dP_{i+2}}{d\theta}(C_{i+2}Y_{i+2} + D_{i+2}Z_{i+2}) = 0.\end{aligned} \tag{3}$$

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*See Love's Treatise, vol. I, p. 318.

The third surface condition, as there is symmetry round $\theta = 0$, is of course identically satisfied.

In dealing with (2) and (3) we require the values of P_2P_i and $\frac{dP_2}{d\theta}\frac{dP_i}{d\theta}$ in terms of spherical harmonics, and the values of $P_2\frac{dP_i}{d\theta}$, $P_i\frac{dP_2}{d\theta}$ and $\frac{dP_2}{d\theta}\frac{d^2P_i}{d\theta^2}$ in terms of differential coefficients of spherical harmonics, as given in equations (30) to (34), p. 306, vol. 16 of this journal. Employing these, we deduce from (2), by equating separately to zero the coefficients of P_{i-2}, P_i and P_{i+2}, the following three equations:

$$A_{i-2}Y_{i-2}+B_{i-2}Z_{i-2} = \frac{3\varepsilon_2 i(i-1)}{(2i-1)(2i+1)}\left\{-\tfrac{1}{2}a\frac{d}{da}(A_iY_i+B_iZ_i)+(i+1)(C_iY_i+D_iZ_i)\right\}, \quad (4)$$

$$A_iY_i+B_iZ_i+\varepsilon_2\frac{i(i+1)}{(2i-1)(2i+3)}\left\{a\frac{d}{da}(A_iY_i+B_iZ_i)-3(C_iY_i+D_iZ_i)\right\}=0, \quad (5)$$

$$A_{i+2}Y_{i+2}+B_{i+2}Z_{i+2} = -\frac{3\varepsilon_2(i+1)(i+2)}{(2i+1)(2i+3)}\left\{\tfrac{1}{2}a\frac{d}{da}(A_iY_i+B_iZ_i)+i(C_iY_i+D_iZ_i)\right\}. \quad (6)$$

Similarly treating (3) and equating separately to zero the coefficients of $\frac{dP_{i-2}}{d\theta}$, $\frac{dP_i}{d\theta}$ and $\frac{dP_{i+2}}{d\theta}$, we get

$$C_{i-2}Y_{i-2}+D_{i-2}Z_{i-2} = \frac{3\varepsilon_2 i}{(2i-1)(2i+1)}\Big[-\tfrac{1}{2}(i+1)a\frac{d}{da}(C_iY_i+D_iZ_i)-(E_iY_i+F_iZ_i) + (i+1)^2(G_iY_i+H_iZ_i)\Big], \quad (7)$$

$$C_iY_i+D_iZ_i+\frac{\varepsilon_2}{(2i-1)(2i+3)}\Big[(i^2+i-3)a\frac{d}{da}(C_iY_i+D_iZ_i) -3(E_iY_i+F_iZ_i)-3(i^2+i-3)(G_iY_i+H_iZ_i)\Big]=0, \quad (8)$$

$$C_{i+2}Y_{i+2}+D_{i+2}Z_{i+2} = \frac{3\varepsilon_2(i+1)}{(2i+1)(2i+3)}\Big[-\tfrac{1}{2}ia\frac{d}{da}(C_iY_i+D_iZ_i)+E_iY_i+F_iZ_i - i^2(G_iY_i+H_iZ_i)\Big]. \quad (9)$$

§18. We see from (5) and (8) that in a perfect sphere the frequency equation arises from the elimination of Y_i and Z_i between the two equations

$$A_i Y_i + B_i Z_i = 0, \tag{10}$$

$$C_i Y_i + D_i Z_i = 0, \tag{11}$$

and so is

$$A_i D_i - B_i C_i = 0. \tag{12}$$

This supplies of course the first approximations to the values of k in the present problem.

Use may be made of (10), (11) and (12) to simplify any terms containing ε_2; in particular, $\varepsilon_2 (C_i Y_i + D_i Z_i)$ may be neglected in (4), (5) and (6).

We may obviously determine in terms of Y_i the constants Y_{i-2}, Z_{i-2} from (4) and (7), the constants Y_{i+2}, Z_{i+2} from (6) and (9).

For the frequency equation we eliminate Y_i and Z_i between (5) and (8), obtaining on reduction

$$\begin{aligned} A_i D_i - B_i C_i + \frac{\varepsilon_2}{(2i-1)(2i+3)} \Big[i(i+1)\Big(D_i a \frac{dA_i}{da} - C_i a \frac{dB_i}{da}\Big) \\ + (i^2+i-3)\Big(A_i a \frac{dD_i}{da} - B_i a \frac{dC_i}{da}\Big) - 3(A_i F_i - B_i E_i) \\ - 3(i^2+i-3)(A_i H_i - B_i G_i)\Big] = 0. \end{aligned} \tag{13}$$

Reasoning from (12) and (13) as before, we deduce for the increment $\delta k / 2\pi$ in pitch, due to the departure of (1) from the truly spherical form, the result

$$\begin{aligned} \frac{\delta k}{k} = \frac{-3\varepsilon_2}{(2i-1)(2i+3)} \Bigg[\frac{i(i+1)}{3} \\ - \frac{A_i a \dfrac{dD_i}{da} - B_i a \dfrac{dC_i}{da} + A_i F_i - B_i E_i + (i^2+i-3)(A_i H_i - B_i G_i)}{a \dfrac{d}{da}(A_i D_i - B_i C_i)} \Bigg]. \end{aligned} \tag{14}$$

Referring to (27), (28) and (29), Section I, and making use of (12), we find on reduction

$$\begin{aligned} \frac{\delta k}{k} = \frac{-3\varepsilon_2}{(2i-1)(2i+3)} \Bigg[\frac{i(i+1)}{3} \\ - \frac{y^2\{y^2 - 4(i-1)(i+2)\} A_i^2}{y^2\{y^2-(i-1)(i+2)\} A_i^2 + i(i+1)\, y^4 C_i^2 - i(i+1)(3y^2 - 4x^2)(A_i + 2C_i)^2} \Bigg]. \end{aligned} \tag{15}$$

In (15) we are to substitute for x and y the values they possess for the root of (12) under consideration; the value of m/n, and so of x/y, being supposed of course known.

§19. For most values of m/n the determination of the roots of (12), and of the corresponding values of A_i/C_i to be used in (15), would unquestionably be tedious. The special case of a nearly incompressible material of finite rigidity is however comparatively simple. For it, using (30) and (31) of Section I, we obtain at once from (15)

$$\frac{\delta k}{k}=\frac{-3\epsilon_2}{(2i-1)(2i+3)}\left[\frac{i(i+1)}{3}-\frac{\{y^2-2i(i-1)\}^2\{y^2-4(i-1)(i+2)\}}{\{y^6-2y^4(2i-1)(2i+1)+12y^2i(i-1)(i+1)(2i+1)-8i(i-1)^2(i+2)(2i^2+4i+3)\}}\right]. \quad (16)$$

Here y is a root of

$$\frac{y^4-y^2(2i-1)(2i+1)+2(i-1)i(i+2)(2i+1)}{y^2-2(i-1)i(i+2)}+\frac{2yJ'_{i+\frac{1}{2}}(y)}{J_{i+\frac{1}{2}}(y)}=0, \quad (17)$$

which is the form taken by (12) when $x/y=0$.

For values of i up to 5 the values of $2yJ'_{i+\frac{1}{2}}(y)/J_{i+\frac{1}{2}}(y)$ are obtained at once from the table on p. 42 of Gray and Mathews' Treatise, and the corresponding frequency equations it will be found are by no means very formidable. Thus for $i=1$, (17) takes the form

$$\tan y=y(y^2-6)\div\{3(y^2-2)\}, \quad (18)$$

and for $i=2$,

$$\tan y=-y(5y^4-92y^2+480)/(y^6-25y^4+252y^2-480). \quad (19)$$

These equations were originally given by Professor Lamb in his paper already referred to (Proc. London Math. Soc., vol. 13), and in the same source will be found the approximate numerical values of the lower roots, viz. for (18),

$$y/\pi=1{\cdot}2319,\ 2{\cdot}3692,\ 3{\cdot}4101,\ 4{\cdot}4310,\ 5{\cdot}4439,\ 6{\cdot}4528,$$

and for (19),

$$y/\pi=0{\cdot}8485,\ 1{\cdot}7420,\ 2{\cdot}8257,\ 3{\cdot}8709.$$

For $i=1$ we deduce from (16)

$$\frac{\delta k}{k}=\tfrac{1}{5}\epsilon_2\frac{y^2+12}{y^2-6}, \quad (20)$$

and for $i = 2$,*

$$\frac{\delta k}{k} = -\tfrac{1}{7}\varepsilon_2 \frac{y^6 - 36y^4 + 576y^2 - 2176}{y^6 - 30y^4 + 360y^2 - 1216}. \tag{21}$$

Numerical values of $\delta k/k$ are thus easily obtainable in these two cases for the several values of y recorded above.

It is obvious when $i = 1$ that $\delta k/k$ is even for the lowest note of the same sign as ε_2. But when $i = 2$, $\delta k/k$ is always opposite in sign to ε_2.

In the general case of incompressible material, when i is not very large, we deduce from (16) for very high notes—i. e. for large values of y—as an approximation

$$\delta k/k = -\varepsilon_2 (i^2 + i - 3) \div \{(2i-1)(2i+3)\}. \tag{22}$$

Here when i exceeds 1, the sign of $\delta k/k$ is opposite to that of ε_2, or the pitch is raised or lowered according as the spheroid is oblate or prolate.

The identity of (22) with the corresponding result, (21), Section III, for transverse vibrations is noteworthy.

SECTION VI.

§20. The case of approximately radial and transverse vibrations depending on P_2 in a spheroid presents a certain peculiarity inasmuch as P_{2-2} or P_0 is a constant. I do not propose, however, to consider this case specially, but pass on to a more general case of an ellipsoid which illustrates equally this peculiarity.

Use will be made of the notation

$$P_{i,s}(\mu) = \frac{\left|\underline{i-s}\right.}{1.3 \ldots . (2i-1)} (1-\mu^2)^{\frac{s}{2}} \frac{d^s}{d\mu^s} P_i(\mu), \tag{1}$$

where $\mu \equiv \cos\theta$.

Also for brevity let

$$P_{2,2} \cos 2\phi \equiv \sin^2\theta \cos 2\phi = X_{2,2}. \tag{2}$$

I propose to consider the vibrations which in a perfect sphere would be of the type radial and transverse, involving the harmonics P_2 and $X_{2,2}$, taking place in the nearly spherical solid contained by

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* In strictness the application of (16) when $i = 2$ is not fully warranted at the present stage, see §20; it is however justified by §23, and for convenient comparison with the case $i = 1$ it is introduced here.

$$r = a(1 + \varepsilon P_2 + \varepsilon_2 X_{2,2}). \tag{3}$$

Squares and products of ε and ε_2 will be neglected. The lengths a', b', c' of the semi-axes of the ellipsoid (3) are given by

$$\left.\begin{aligned} a'/a &= 1 - \tfrac{1}{2}\varepsilon + \varepsilon_2, \\ b'/a &= 1 - \tfrac{1}{2}\varepsilon - \varepsilon_2, \\ c'/a &= 1 + \varepsilon. \end{aligned}\right\} \tag{4}$$

The surface conditions take the form

$$\left.\begin{aligned} \widehat{rr} - \frac{d}{d\theta}(\varepsilon P_2 + \varepsilon_2 X_{2,2})\,\widehat{r\theta} - \frac{1}{\sin\theta}\frac{d}{d\phi}(\varepsilon_2 X_{2,2})\,\widehat{r\phi} &= 0, \\ \widehat{r\theta} - \frac{d}{d\theta}(\varepsilon P_2 + \varepsilon_2 X_{2,2})\,\widehat{\theta\theta} - \frac{1}{\sin\theta}\frac{d}{d\phi}(\varepsilon_2 X_{2,2})\,\widehat{\theta\phi} &= 0, \\ \widehat{r\phi} - \frac{d}{d\theta}(\varepsilon P_2 + \varepsilon_2 X_{2,2})\,\widehat{\theta\phi} - \frac{1}{\sin\theta}\frac{d}{d\phi}(\varepsilon_2 X_{2,2})\,\widehat{\phi\phi} &= 0. \end{aligned}\right\} \tag{5}$$

In addition to principal terms representing radial and transverse vibrations depending on P_2 and $X_{2,2}$, we require subsidiary terms representing pure radial vibrations, subsidiary terms representing radial and transverse vibrations depending on P_4, $P_{4,2}\cos 2\phi$ and $P_{4,4}\cos 4\phi$, and lastly, subsidiary terms representing pure transverse vibrations depending on $P_{3,2}\sin 2\phi$. The complete values of the displacements are as follows:

$$\begin{aligned} u/\cos kt = \frac{r^{-\frac{3}{2}}}{2k^2\alpha^2}&\left[\{J_{\frac{1}{2}}(k\alpha r) - 2k\alpha r J'_{\frac{1}{2}}(k\alpha r)\}\,Y_0\right. \\ &+ \{J_{\frac{5}{2}}(k\alpha r) - 2k\alpha r J'_{\frac{5}{2}}(k\alpha r)\}(Y_2 P_2 + Y_{2,2} X_{2,2}) \\ &\left.+ \{J_{\frac{9}{2}}(k\alpha r) - 2k\alpha r J'_{\frac{9}{2}}(k\alpha r)\}(Y_4 P_4 + Y_{4,2} P_{4,2}\cos 2\phi + Y_{4,4} P_{4,4}\cos 4\phi)\right] \\ &+ r^{-\frac{3}{2}}\left[J_{\frac{5}{2}}(k\beta r)(Z_2 P_2 + Z_{2,2} X_{2,2})\right. \\ &\left.+ J_{\frac{9}{2}}(k\beta r)(Z_4 P_4 + Z_{4,2} P_{4,2}\cos 2\phi + Z_{4,4} P_{4,4}\cos 4\phi)\right], \end{aligned} \tag{6}$$

$$\begin{aligned} v/\cos kt = & -\frac{r^{-\frac{3}{2}}}{k^2\alpha^2}\frac{d}{d\theta}\,[J_{\frac{5}{2}}(k\alpha r)(Y_2P_2 + Y_{2,\,2}X_{2,\,2}) \\ & + J_{\frac{9}{2}}(k\alpha r)(Y_4P_4 + Y_{4,\,2}P_{4,\,2}\cos 2\phi + Y_{4,\,4}P_{4,\,4}\cos 4\phi)] \\ & + \frac{1}{r}\frac{d^2}{d\theta dr}\,r^{\frac{1}{2}}\,\{\tfrac{1}{6}J_{\frac{5}{2}}(k\beta r)(Z_2P_2 + Z_{2,\,2}X_{2,\,2}) \\ & + \tfrac{1}{20}J_{\frac{9}{2}}(k\beta r)(Z_4P_4 + Z_{4,\,2}P_{4,\,2}\cos 2\phi + Z_{4,\,4}P_{4,\,4}\cos 4\phi)\} \\ & + r^{-\frac{1}{2}}\frac{1}{\sin\theta}\frac{d}{d\phi}\,\{W_{3,\,2}P_{3,\,2}\sin 2\phi\, J_{\frac{7}{2}}(k\beta r)\}, \end{aligned} \tag{7}$$

$$\begin{aligned} w/\cos kt = & -\frac{r^{-\frac{3}{2}}}{k^2\alpha^2}\frac{1}{\sin\theta}\frac{d}{d\phi}\,[J_{\frac{5}{2}}(k\alpha r)\,Y_{2,\,2}X_{2,\,2} \\ & + J_{\frac{9}{2}}(k\alpha r)(Y_{4,\,2}P_{4,\,2}\cos 2\phi + Y_{4,\,4}P_{4,\,4}\cos 4\phi)] \\ & + \frac{1}{r\sin\theta}\frac{d^2}{d\phi\, dr}\,r^{\frac{1}{2}}\,\{\tfrac{1}{6}J_{\frac{5}{2}}(k\beta r)\,Z_{2,\,2}X_{2,\,2} \\ & + \tfrac{1}{20}J_{\frac{9}{2}}(k\beta r)(Z_{4,\,2}P_{4,\,2}\cos 2\phi + Z_{4,\,4}P_{4,\,4}\cos 4\phi)] \\ & - r^{-\frac{1}{2}}\frac{d}{d\theta}\,\{W_{3,\,2}P_{3,\,2}\sin 2\phi J_{\frac{7}{2}}(k\beta r)\}. \end{aligned} \tag{8}$$

§21. The several letters Y, Z, W represent constants, the ratio between which will be determined from the surface conditions. The suffixes serve conveniently to show in the subsequent work the origin of the several terms.

Referring to equations (8) to (13) and (37) of Section I, and employing our previous notation, we find the surface conditions are as follows:

$$\begin{aligned} A_0Y_0 + & \left\{1 + (\varepsilon P_2 + \varepsilon_2 X_{2,\,2})\, a\frac{d}{da}\right\}[(A_2Y_2 + B_2Z_2)\,P_2 + (A_2Y_{2,\,2} + B_2Z_{2,\,2})\,X_{2,\,2}] \\ & - \{\varepsilon(C_2Y_{2,\,2} + D_2Z_{2,\,2}) + \varepsilon_2(C_2Y_2 + D_2Z_2)\}\frac{dP_2}{d\theta}\frac{dX_{2,\,2}}{d\theta} \\ & - \varepsilon(C_2Y_2 + D_2Z_2)\left(\frac{dP_2}{d\theta}\right)^2 \\ & - \varepsilon_2(C_2Y_{2,\,2} + D_2Z_{2,\,2})\left\{\left(\frac{d}{d\theta}X_{2,\,2}\right)^2 + \left(\frac{1}{\sin\theta}\frac{dX_{2,\,2}}{d\phi}\right)^2\right\} + (A_4Y_4 + B_4Z_4)\,P_4 \\ & + (A_4Y_{4,\,2} + B_4Z_{4,\,2})\,P_{4,\,2}\cos 2\phi + (A_4Y_{4,\,4} + B_4Z_{4,\,4})\,P_{4,\,4}\cos 4\phi = 0, \end{aligned} \tag{9}$$

$$
\begin{aligned}
&\left\{1 + (\varepsilon P_2 + \varepsilon_2 X_{2,\,2})\, a \frac{d}{da}\right\}\left\{(C_2 Y_2 + D_2 Z_2)\frac{dP_2}{d\theta} + (C_2 Y_{2,\,2} + D_2 Z_{2,\,2})\frac{dX_{2,\,2}}{d\theta}\right\} \\
&\quad - \frac{d}{d\theta}(\varepsilon P_2 + \varepsilon_2 X_{2,\,2})\Big[(E_2 Y_2 + F_2 Z_2)\,P_2 + (E_2 Y_{2,\,2} + F_2 Z_{2,\,2})\,X_{2,\,2} \\
&\quad + \frac{d^2}{d\theta^2}(G_2 Y_2 + H_2 Z_2)\,P_2 + \frac{d^2}{d\theta^2}(G_2 Y_{2,\,2} + H_2 Z_{2,\,2})\,X_{2,\,2}\Big] \\
&\quad - \varepsilon_2 \frac{1}{\sin\theta}\frac{dX_{2,\,2}}{d\phi}\frac{d^2}{d\theta d\phi}(G_2 Y_{2,\,2} + H_2 Z_{2,\,2}) \operatorname{cosec}\theta X_{2,\,2} \\
&\quad + \frac{d}{d\theta}[(C_4 Y_4 + D_4 Z_4)\,P_4 + (C_4 Y_{4,\,2} + D_4 Z_{4,\,2})\,P_{4,\,2}\cos 2\phi \\
&\quad + (C_4 Y_{4,\,4} + D_4 Z_{4,\,4})\,P_{4,\,4}\cos 4\phi] \\
&\quad + \frac{1}{\sin\theta}\frac{d}{d\phi}\{\tfrac{1}{2}\, a B_3 W_{3,\,2} P_{3,\,2}\sin 2\phi\} = 0, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
&\left\{1 + (\varepsilon P_2 + \varepsilon_2 X_{2,\,2})\, a \frac{d}{da}\right\}\frac{1}{\sin\theta}\frac{d}{d\phi}(C_2 Y_{2,\,2} + D_2 Z_{2,\,2})\,X_{2,\,2} \\
&\quad - \frac{d}{d\theta}(\varepsilon P_2 + \varepsilon_2 X_{2,\,2})\frac{d^2}{d\theta d\phi}(G_2 Y_{2,\,2} + H_2 Z_{2,\,2}) \operatorname{cosec}\theta X_{2,\,2} \\
&\quad - \varepsilon_2 \frac{1}{\sin\theta}\frac{dX_{2,\,2}}{d\phi}\Big[(E_2 Y_2 + F_2 Z_2)\,P_2 + \cot\theta \frac{d}{d\theta}(G_2 Y_2 + H_2 Z_2)\,P_2 \\
&\quad + (E_2 Y_{2,\,2} + F_2 Z_{2,\,2})\,X_{2,\,2} \\
&\quad + \left(\frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2} + \cot\theta\frac{d}{d\theta}\right)(G_2 Y_{2,\,2} + H_2 Z_{2,\,2})\,X_{2,\,2}\Big] \\
&\quad + \frac{1}{\sin\theta}\frac{d}{d\phi}[(C_4 Y_{4,\,2} + D_4 Z_{4,\,2})\,P_{4,\,2}\cos 2\phi + (C_4 Y_{4,\,4} + D_4 Z_{4,\,4})\,P_{4,\,4}\cos 4\phi] \\
&\quad - \frac{d}{d\theta}\{\tfrac{1}{2} a B_3 W_{3,\,2} P_{3,\,2}\sin 2\phi\}. \qquad (11)
\end{aligned}
$$

In the first surface condition we have to express each term as a constant or a sum of spherical harmonics; in the second and third surface conditions each term must be expressed in terms of differential coefficients of spherical harmonics. The requisite results, if not all explicitly given on pp. 327–331, vol. 16 of this journal, are immediately deducible from the results there given. They consist of the expression of $(P_2)^2$, $P_2 X_{2,\,2}$, $(X_{2,\,2})^2$, $\left(\frac{dP_2}{d\theta}\right)^2$, $\left(\frac{dX_{2,\,2}}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{dX_{2,\,2}}{d\phi}\right)^2$, and

$\frac{dP_2}{d\theta}\ \frac{dX_{2,2}}{d\theta}$ as sums of constants and spherical harmonics; the expression of

$P_2\frac{dP_2}{d\theta}$, $P_2\frac{dX_{2,2}}{d\theta}$, $X_{2,2}\frac{dP_2}{d\theta}$, $X_{2,2}\frac{dX_{2,2}}{d\theta}$, $\frac{dP_2}{d\theta}\ \frac{d^2P_{2,2}}{d\theta^2}$, $\frac{dP_2}{d\theta}\ \frac{d^2X_{2,2}}{d\theta^2}$, $\frac{dX_{2,2}}{d\theta}\ \frac{d^2P_2}{d\theta^2}$

and $\frac{dX_{2,2}}{d\theta}\ \frac{d^2X_{2,2}}{d\theta^2}+\frac{1}{\sin\theta}\ \frac{dX_{2,2}}{d\phi}\ \frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta X_{2,2})$ as differential coefficients with respect to θ of a sum of spherical harmonics; the expression of

$P_2\frac{1}{\sin\theta}\ \frac{dX_{2,2}}{d\phi}$, $\cot\theta\frac{dP_2}{d\theta}\ \frac{1}{\sin\theta}\ \frac{dX_{2,2}}{d\phi}$, $X_{2,2}\frac{1}{\sin\theta}\ \frac{dX_{2,2}}{d\phi}$, $\frac{dP_2}{d\theta}\ \frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta X_{2,2})$

and $\frac{dX_{2,2}}{d\theta}\ \frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta X_{2,2})+\frac{1}{\sin\theta}\ \frac{dX_{2,2}}{d\phi}\left(\frac{1}{\sin^2\theta}\ \frac{d^2}{d\phi^2}+\cot\theta\frac{d}{d\theta}\right)X_{2,2}$

as differentials with respect to ϕ of a sum of spherical harmonics divided by $\sin\theta$.

It appears unnecessary to reproduce the results.

Use must also be made of the identities

$$\frac{1}{\sin\theta}\ \frac{d}{d\phi}(P_{3,2}\sin 2\phi)=\frac{d}{d\theta}(P_{2,2}\cos 2\phi),$$

$$\frac{d}{d\theta}(P_{3,2}\sin 2\phi)=\frac{1}{\sin\theta}\ \frac{d}{d\phi}\left(\tfrac{2}{7}P_{2,2}-\tfrac{3}{2}P_{4,2}\right)\cos 2\phi.$$

§22. Exactly the same method of procedure is to be followed as in the case of equilibrium, so it will suffice to record the resulting equations. These are as follows:

$$Y_0=-\frac{1}{5A_0}a\frac{d}{da}\left[\varepsilon(A_2Y_2+B_2Z_2)+\tfrac{4}{3}\varepsilon_2(A_2Y_{2,2}+B_2Z_{2,2})\right], \qquad (12)$$

$$\left.\begin{aligned}A_2Y_2+B_2Z_2&=-\tfrac{2}{7}a\frac{d}{da}\left[\varepsilon(A_2Y_2+B_2Z_2)-\tfrac{4}{3}\varepsilon_2(A_2Y_{2,2}+B_2Z_{2,2})\right],\\ C_2Y_2+D_2Z_2&=-\tfrac{1}{7}\Big[\varepsilon\Big\{a\frac{d}{da}(C_2Y_2+D_2Z_2)-(E_2Y_2+F_2Z_2)\\ &\quad-3(G_2Y_2+H_2Z_2)\Big\}-\tfrac{4}{3}\varepsilon_2\Big\{a\frac{d}{da}(C_2Y_{2,2}+D_2Z_{2,2})\\ &\quad-(E_2Y_{2,2}+F_2Z_{2,2})-3(G_2Y_{2,2}+H_2Z_{2,2})\Big\}\Big],\end{aligned}\right\}\qquad (13)$$

$$\left.\begin{aligned}
A_2Y_{2,2}+B_2Z_{2,2} &= \tfrac{2}{7}\,a\frac{d}{da}\left[\varepsilon\left(A_2Y_{2,2}+B_2Z_{2,2}\right)+\varepsilon_2\left(A_2Y_2+B_2Z_2\right)\right],\\
C_2Y_{2,2}+D_2Z_{2,2} &= \tfrac{1}{7}\Big[\varepsilon\Big\{a\frac{d}{da}\left(C_2Y_{2,2}+D_2Z_{2,2}\right)\\
&\quad -\left(E_2Y_{2,2}+F_2Z_{2,2}\right)-3\left(G_2Y_{2,2}+H_2Z_{2,2}\right)\Big\}\\
&\quad +\varepsilon_2\Big\{a\frac{d}{da}\left(C_2Y_2+D_2Z_2\right)-\left(E_2Y_2+F_2Z_2\right)\\
&\quad -3\left(G_2Y_2+H_2Z_2\right)\Big\}\Big],
\end{aligned}\right\}\quad(14)$$

$$\left.\begin{aligned}
A_4Y_4+B_4Z_4 &= -\tfrac{2}{35}\,a\frac{d}{da}\left[9\varepsilon\left(A_2Y_2+B_2Z_2\right)+2\varepsilon_2\left(A_2Y_{2,2}+B_2Z_{2,2}\right)\right],\\
C_4Y_4+D_4Z_4 &= -\tfrac{1}{35}\Big[9\varepsilon\Big\{a\frac{d}{da}\left(C_2Y_2+D_2Z_2\right)\\
&\quad -\left(E_2Y_2+F_2Z_2\right)+4\left(G_2Y_2+H_2Z_2\right)\Big\}\\
&\quad +2\varepsilon_2\Big\{a\frac{d}{da}\left(C_2Y_{2,2}+D_2Z_{2,2}\right)\\
&\quad -\left(E_2Y_{2,2}+F_2Z_{2,2}\right)+4\left(G_2Y_{2,2}+H_2Z_{2,2}\right)\Big\}\Big],
\end{aligned}\right\}\quad(15)$$

$$\left.\begin{aligned}
A_4Y_{4,2}+B_4Z_{4,2} &= -\tfrac{3}{2}\,a\frac{d}{da}\left[\varepsilon\left(A_2Y_{2,2}+B_2Z_{2,2}\right)+\varepsilon_2\left(A_2Y_2+B_2Z_2\right)\right],\\
C_4Y_{4,2}+D_4Z_{4,2} &= -\tfrac{3}{4}\Big[\varepsilon\Big\{a\frac{d}{da}\left(C_2Y_{2,2}+D_2Z_{2,2}\right)\\
&\quad -\left(E_2Y_{2,2}+F_2Z_{2,2}\right)+4\left(G_2Y_{2,2}+H_2Z_{2,2}\right)\Big\}\\
&\quad +\varepsilon_2\Big\{a\frac{d}{da}\left(C_2Y_2+D_2Z_2\right)\\
&\quad -\left(E_2Y_2+F_2Z_2\right)+4\left(G_2Y_2+H_2Z_2\right)\Big\}\Big],
\end{aligned}\right\}\quad(16)$$

$$\left.\begin{aligned}
A_4Y_{4,4}+B_4Z_{4,4} &= -\tfrac{1}{2}\,\varepsilon_2 a\frac{d}{da}\left(A_2Y_{2,2}+B_2Z_{2,2}\right),\\
C_4Y_{4,4}+D_4Z_{4,4} &= -\tfrac{1}{4}\,\varepsilon_2\Big\{a\frac{d}{da}\left(C_2Y_{2,2}+D_2Z_{2,2}\right)\\
&\quad -\left(E_2Y_{2,2}+F_2Z_{2,2}\right)+4\left(G_2Y_{2,2}+H_2Z_{2,2}\right)\Big\},
\end{aligned}\right\}\quad(17)$$

$$\begin{aligned}
W_{3,2}=\frac{1}{aB_3}\Big[-\varepsilon\Big\{a\frac{d}{da}\left(C_2Y_{2,2}+D_2Z_{2,2}\right)+E_2Y_{2,2}+F_2Z_{2,2}\Big\}\\
+\varepsilon_2\Big\{a\frac{d}{da}\left(C_2Y_2+D_2Z_2\right)+\left(E_2Y_2+F_2Z_2\right)\Big\}\Big].\quad(18)
\end{aligned}$$

The three surface conditions are satisfied exactly, to the assumed degree of approximation, when the equations (12) to (18) are satisfied.

As first approximations, when terms in ε and ε_2 are neglected, we obtain from (13) and (14)

$$\left.\begin{aligned} A_2 Y_2 + B_2 Z_2 &= 0, \\ C_2 Y_2 + D_2 Z_2 &= 0, \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} A_2 Y_{2,2} + B_2 Z_{2,2} &= 0, \\ C_2 Y_{2,2} + D_2 Z_{2,2} &= 0. \end{aligned}\right\} \tag{20}$$

Eliminating Y_2/Z_2 from (19), and $Y_{2,2}/Z_{2,2}$ from (20), we arrive in either case at the frequency equation,

$$A_2 D_2 - B_2 C_2 = 0. \tag{21}$$

This supplies, of course, the first approximation to the values of k in the present problem.

In a perfect sphere the constants Y_2, $Y_{2,2}$ are absolutely independent of one another and their ratio may have any value. The vibrations answering to P_2 and those answering to $X_{2,2}$, though possessing the same frequency, may exist side by side or separately, uninfluenced the one by the other.

§23. The same independence is exhibited when the ellipsoid (3) is a spheroid. For putting $\varepsilon_2 = 0$ in (13) and (14) we get two completely independent sets of equations, viz.

$$\left.\begin{aligned} A_2 Y_2 + B_2 Z_2 &= -\tfrac{2}{7}\varepsilon a \frac{d}{da}(A_2 Y_2 + B_2 Z_2), \\ C_2 Y_2 + D_2 Z_2 &= -\tfrac{1}{7}\varepsilon \Big[a \frac{d}{da}(C_2 Y_2 + D_2 Z_2) \\ &\qquad - (E_2 Y_2 + F_2 Z_2) - 3(G_2 Y_2 + H_2 Z_2)\Big] \end{aligned}\right\} \tag{22}$$

and

$$\left.\begin{aligned} A_2 Y_{2,2} + B_2 Z_{2,2} &= \tfrac{2}{7}\varepsilon a \frac{d}{da}(A_2 Y_{2,2} + B_2 Z_{2,2}), \\ C_2 Y_{2,2} + D_2 Z_{2,2} &= \tfrac{1}{7}\varepsilon \Big[a \frac{d}{da}(C_2 Y_{2,2} + D_2 Z_{2,2}) \\ &\qquad - (E_2 Y_{2,2} + F_2 Z_{2,2}) - 3(G_2 Y_{2,2} + H_2 Z_{2,2})\Big]. \end{aligned}\right\} \tag{23}$$

Proceeding in the same manner as in previous cases, we find for the increments of pitch in the spheroid, relative to the pitch in the perfect sphere, from

(22) and (23) respectively

$$\delta k_1/k = -N\varepsilon, \tag{24}$$

$$\delta k_2/k = +N\varepsilon, \tag{25}$$

where

$$N = \tfrac{1}{7}\left[2 - \frac{A_2 a\dfrac{dD_2}{da} - B_2 a\dfrac{dC_2}{da} + (A_2F_2 - B_2E_2) + 3\,(A_2H_2 - B_2G_2)}{a\dfrac{d}{da}\,(A_2D_2 - B_2C_2)}\right]. \tag{26}$$

The identity of $-N$ with the coefficient of ε_2 in (14), Section V, when 2 is written for i, should be noticed.

It will be observed that δk_1 refers to the vibration depending on P_2, δk_2 to that depending on $X_{2,\,2}$.

§24. In the general case of the ellipsoid, the independence of the vibrations depending on P_2 and on $X_{2,\,2}$ ceases. For equations (13) and (14) lead to

$$Y_{2,\,2}/Y_2 = Z_{2,\,2}/Z_2 = q, \tag{27}$$

where

$$q\varepsilon + \varepsilon_2 = q\,(-\varepsilon + \tfrac{4}{3}\,q\varepsilon_2), \tag{28}$$

or

$$q = \tfrac{3}{4}\left\{\frac{\varepsilon}{\varepsilon_2} \mp \sqrt{\left(\frac{\varepsilon}{\varepsilon_2}\right)^2 + \tfrac{4}{3}}\right\}. \tag{29}$$

Employing (27) in either (13) or (14), we get

$$\left.\begin{aligned} A_2Y_2 + B_2Z_2 &= \tfrac{2}{7}\,\frac{q\varepsilon + \varepsilon_2}{q}\;a\frac{d}{da}\,(A_2Y_2 + B_2Z_2),\\ C_2Y_2 + D_2Z_2 &= \tfrac{1}{7}\,\frac{q\varepsilon + \varepsilon_2}{q}\Big[a\frac{d}{da}\,(C_2Y_2 + D_2Z_2)\\ &\qquad - (E_2Y_2 + F_2Z_2) - 3\,(G_2Y_2 + H_2Z_2)\Big].\end{aligned}\right\} \tag{30}$$

Comparing (30) with (23), we deduce at once for the change of frequency due to the departure of the surface from the truly spherical form

$$\begin{aligned}\delta k/k &= N(q\varepsilon + \varepsilon_2)/q\\ &= \mp N\sqrt{\varepsilon^2 + \tfrac{4}{3}\varepsilon_2^2}\end{aligned} \tag{31}$$

by (29), where N is given by (26).

The interpretation to be put upon the result is that in the general case of the ellipsoid (3), the principal terms in P_2 and $X_{2,\,2}$ do not occur independently but stand to one another in one of two definite ratios given by the values of q in

(29). In one of the cases the pitch is raised relative to what it would be in the perfect sphere, in the other it is lowered by an equal amount.

By means of (4) we may exhibit (31) in the more symmetrical form

$$\frac{\delta k}{k} = \mp 2N\{1 - 3(a'b' + a'c' + b'c')/(a' + b' + c')^2\}^{\frac{1}{2}}. \tag{32}$$

§25. The values of the constants appearing in the subsidiary terms are derived from (12) and (15) to (18), use being made of (19), (20) and (21).

For shortness let

$$\mu_2 = \frac{(Y_2/B_2)}{140(A_4D_4 - B_4C_4)}\Big[-B_4\Big\{A_2a\frac{dD_2}{da} - B_2a\frac{dC_2}{da} - (A_2F_2 - B_2E_2) \\ + 4(A_2H_2 - B_2G_2)\Big\} + 2D_4\Big(A_2a\frac{dB_2}{da} - B_2a\frac{dA_2}{da}\Big)\Big], \tag{33}$$

$$\mu_2' = \frac{(Y_2/B_2)}{140(A_4D_4 - B_4C_4)}\Big[A_4\Big\{A_2a\frac{dD_2}{da} - B_2a\frac{dC_2}{da} - (A_2F_2 - B_2E_2) \\ + 4(A_2H_2 - B_2G_2)\Big\} - 2C_4\Big(A_2a\frac{dB_2}{da} - B_2a\frac{dA_2}{da}\Big)\Big]. \tag{34}$$

Then we get

$$Y_0 = -\frac{(\varepsilon + \frac{4}{3}q\varepsilon_2)Y_2}{5A_0B_2}\Big(B_2a\frac{dA_2}{da} - A_2a\frac{dB_2}{da}\Big), \tag{35}$$

$$W_{3,2} = \frac{(q\varepsilon - \varepsilon_2)Y_2}{aB_2B_3}\Big\{A_2a\frac{dD_2}{da} - B_2a\frac{dC_2}{da} + (A_2F_2 - B_2E_2)\Big\}, \tag{36}$$

$$Y_4/\mu_2 = Z_4/\mu_2' = 36\varepsilon + 8q\varepsilon_2, \tag{37}$$

$$Y_{4,2}/\mu_2 = Z_{4,2}/\mu_2' = 105(q\varepsilon + \varepsilon_2), \tag{38}$$

$$Y_{4,4}/\mu_2 = Z_{4,4}/\mu_2' = 35q\varepsilon_2. \tag{39}$$

To simplify the results, we may make use of (21) and of the results (18), (19), etc., of Section I. We in this way reduce (35) and (36) to

$$Y_0 = -\frac{(\varepsilon + \frac{4}{3}q\varepsilon_2)C_2Y_2}{5A_0}\,\frac{(3y^2 - 4x^2)(2 + A_2/C_2) - \frac{1}{2}y^4}{y^2 - 8}, \tag{40}$$

$$W_{3,2} = -\frac{(q\varepsilon - \varepsilon_2)A_2Y_2}{12aB_3}\,\frac{y^2(y^2 - 10)}{y^2 - 8}. \tag{41}$$

Referring to (29) it will be found that Y_0 vanishes when either

$$\varepsilon_2 = 0 \text{ and } Y_2 = 0,$$

or when

$$2\varepsilon_2 \pm 3\varepsilon = 0, \tag{42}$$

and the proper sign is taken in the value of q.

The first alternative answers to a spheroid whose axis is $\theta = 0$, the vibrations which require no subsidiary pure radial term being those which depend on the harmonic $P_{2,\,2} \cos 2\phi$.

The equation (42) is satisfied when

$$c' = b'$$

or

$$c' = a';$$

and so likewise implies the ellipsoid being a surface of revolution. When $c' = b'$ we have

$$q = \tfrac{1}{2},$$

and the harmonic occurring in the displacements is

$$P_2 + \tfrac{1}{2} P_{22} \cos 2\phi \equiv \cos^2\theta - \sin^2\theta \sin^2\phi;$$

when $c' = a'$ we have

$$q = -\tfrac{1}{2},$$

and the harmonic occurring in the displacements is

$$P_2 - \tfrac{1}{2} P_{22} \cos 2\phi \equiv \cos^2\theta - \sin^2\theta \cos^2\phi.$$

If for a moment we suppose rectangular coordinates x, y, z coinciding with the principal axes of the ellipsoid, then all three cases are included in the following statement: If $z = 0$ be an axis of symmetry, then no subsidiary pure radial displacement is required if the surface harmonic in the principal term of the radial component of the displacement is $(x^2 - y^2)/r^2$.

In a somewhat similar fashion it is easy to prove from (41) and (29) that $W_{3,\,2}$ vanishes when the ellipsoid is one of revolution, the harmonic appearing in the principal term of the radial displacement having in this case the axis of symmetry for its axis; if, for instance, $\theta = 0$ is the axis of symmetry, the harmonic is P_2.

The close analogy of the results (37), (38), (39) to the corresponding results for the equilibrium problem—vol. 16, equations (263), (266) and (270), p. 377—is noteworthy. In fact, putting

$$\varepsilon_1 = \alpha_2 = \beta_2 = 0,$$

$$\mu_{22}/\mu_2 = \mu'_{22}/\mu'_2 = q,$$

in the equilibrium results, we make them identical in form with (37), (38) and (39).